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On the Singularity of Influence Coefficients of the Externally Pressurized Spherical Shells

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1. Introduction

THE influence coefficients of pressurized spherical shells have been obtained by W. Nachbar,[1] G. B. Cline[2] and by D. Bushnell.[3] The numerical results for the externally pressurized spherical shells calculated by Bushnell show that the influence coefficients have singularity near one-half the classical buckling pressure, whereas the stiffness coefficients behave regular and nonzero. This indicates that the shell becomes softer as the external pressure increases and it cannot carry any load when the pressure reaches the value corresponding to the singularity. A question may then arise; why are the stiffness coefficients not zero? It is the intention of the present Note to clarify the question. It turns out that the singularity corresponds to the buckling at the edge of conical shells whose edge slides and rotates freely on the constraining surface of the conical shape. The conclusion may provide a supplementary explanation to the numerical result obtained recently by Baruch, Harari and Singer.[4]

2. Influence Coefficients

The basic nonlinear equations governing axisymmetric deformations of elastic thin shells of revolution have been formulated by E. Reissner.[5] The dependent variables may be represented as a result of the superposition of quantities belonging to the state of membrane stress and to that of bending stress. The equations are linearized with respect to the quantities belonging to the state of bending—taking the effect of the membrane stresses into account. Solution of the linearized equations yields the following expressions for the transverse shear resultant Q, the change of meridian tangent β, the horizontal component of stress resultant H, the meridional moment M_ϕ, and for the horizontal displacement u:

$$Q = (Eh)(R_1 A + I_1 B)\Psi \tag{1}$$

$$\beta = \lambda_0^2(\gamma_1 A + \gamma_2 B)\Psi \tag{2}$$

$$H = (Eh/\sin\phi)(S_1 A + S_2 B) \tag{3}$$

$$M_\phi = -(D/a)(S_3 A + S_4 B) \tag{4}$$

$$u = a\sin\phi[(S_1' - \cot\phi S_1)A + (S_2' - \cot\phi S_2)B] \tag{5}$$

where h is the thickness, a the radius, ϕ the meridional angle measured from the axis of rotation of the shell, E Young's modulus, D the bending rigidity of the shell, R_1 and I_1 are, respectively, the real and imaginary parts of the Bessel function of the first kind of order one, A and B are constants, and the prime indicates differentiation with respect to ϕ.

Here, the following abbreviations have been introduced:

$$\Psi = (\phi/\sin\phi)^{1/2} \tag{6}$$

$$\lambda_0 = [12(1 - \nu^2)]^{1/4}(a/h)^{1/2} \tag{7}$$

$$\gamma_1 = \rho R_1 + (1 - \rho^2)^{1/2} I_1 \tag{8a}$$

$$\gamma_2 = \rho I_1 - (1 - \rho^2)^{1/2} R_1 \tag{8b}$$

$$S_1 = -[(1 - 2\rho^2)R_1 - 2\rho(1 - \rho^2)^{1/2} I_1]\Psi \tag{9a}$$

$$S_2 = -[(1 - 2\rho^2)I_1 + 2\rho(1 - \rho^2)^{1/2} R_1]\Psi \tag{9b}$$

$$S_3 = \lambda_0^2\{\rho[(R_1\Psi)' + \nu\cot\phi(R_1\Psi)] + (1 - \rho^2)^{1/2}[(I_1\Psi)' + \nu\cot\phi(I_1\Psi)]\} \tag{9c}$$

$$S_4 = \lambda_0^2\{\rho[(I_1\Psi)' + \nu\cot\phi(I_1\Psi)] - (1 - \rho^2)^{1/2}[(R_1\Psi)' + \nu\cot\phi(R_1\Psi)]\} \tag{9d}$$

where ρ is the ratio of the applied pressure to the classical buckling pressure of complete spherical shells, and ν is Poisson's ratio.

It should be noted that the approximation

$$V = 0 \tag{10}$$

has been made in the derivation of the aforementioned expressions.

The influence coefficients are now obtained by prescribing the inhomogeneous boundary conditions, $H = H_e$ and $M_\phi = M_{\phi e}$, where the subscript e indicates the values at the edge

$$u_e = C_{11}H_e + C_{12}M_{\phi e}, \quad \beta_e = C_{21}H_e + C_{22}M_{\phi e} \tag{11}$$

The result is identica lto that of Bushnell[3] and it may be written in the form

$$\begin{aligned}
C_{11} &= (a\sin^2\alpha/Eh)(1/\Delta)(S_1'S_4 - S_2'S_3 - \Delta\nu\cot\alpha) \\
C_{12} &= (\lambda_0^4\sin\alpha/Eh)(1/\Delta)(S_1'S_2 - S_2'S_1) \\
C_{21} &= (\lambda_0^2\sin\alpha/Eh)(1/\Delta)(\gamma_1 S_4 - \gamma_2 S_3) \\
C_{22} &= (\lambda_0^6/Eh\alpha)(1/\Delta)(\gamma_1 S_2 - \gamma_2 S_1)
\end{aligned} \tag{12}$$

where

$$\Delta = S_1 S_4 - S_2 S_3 \tag{13}$$

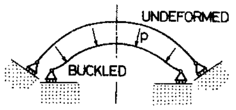

Fig. 1 Boundary condition $H_e = M_{\phi e} = 0$.

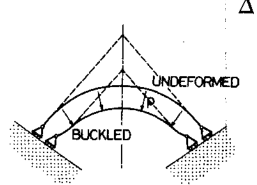

Fig. 2 Boundary condition $Q_{\alpha e} = M_{\phi e} = 0$.

Received February 22, 1971.

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3. Eigenvalues

The eigenvalues for the following six cases of homogeneous boundary conditions are investigated:

Case (*i*) $H_e = 0, \quad M_{\phi e} = 0$ (14)

The characteristic equation takes the form

$$S_1S_4 - S_2S_3 = 0 \tag{15}$$

The left-hand side of Eq. (15) is equal to Δ. The influence coefficients thus become indefinitely large at the value of the pressure satisfying Eq. (15). Therefore, it is concluded that the singularity in the influence coefficients corresponds to the eigenvalue and, consequently, to the buckling pressure of the spherical shell segment supported by the edge condition characterized by Eqs. (14).

The stiffness coefficients $K_{ij}, i, j = 1, 2$, are

$$K_{ij} = \bar{C}_{ij}/(C_{11}C_{22} - C_{12}C_{21}) \tag{16}$$

where $\bar{C}_{ij}$ is the co-factor of C_{ij} in the determinant $|C_{ij}|$.

A simple calculation shows

$$C_{11}C_{22} - C_{12}C_{21} = (\lambda_0^6 \sin^2\alpha/E^2h^2)(1/\Delta)[(\gamma_1S_2' - \gamma_2S_1') - \nu \cot\alpha(\gamma_1S_2 - \gamma_2S_1)] \tag{17}$$

Hence, Δ cancels out and K_{ij} is not necessarily zero when $\Delta = 0$. The result is not surprising at all, because the buckling deformations occur in the eigenmode, and, consequently, u_e and β_e are linearly related and their ratio cannot be changed arbitrarily. Therefore, the stiffness coefficients need not vanish identically.

The boundary condition characterized by Eqs. (14) is quite unrealistic except for hemispherical shells. As sketched in Fig. 1, the shell slides and rotates freely on the constraining surface of conical shape, which becomes horizontal when the buckling occurs. In the case of the hemispherical shell the constraining surface is always horizontal. The eigenvalue $\rho = 0.5$ obtained for the hemispherical shell, therefore, corresponds to the buckling pressure of the hemispherical shell with free edge. Because of the similarity of their behavior near the edge, it can be anticipated that the axially compressed circular cylindrical shell with free edge can buckle at one-half the classical buckling pressure. This is the problem discussed by N. J. Hoff.[6]

Case (*ii*) $Q_{\alpha e} = 0, \quad M_{\phi e} = 0$ (18)

where $Q_{\alpha e}$ is the component of the edge force in the radial direction specified by the edge angle α.

Approximation consistent to Eq. (10) results in

Case $Q_{\alpha e} = -H_e \sin\alpha$ (19)

Therefore, in the present approximation, the boundary condition characterized by Eqs. (18) is identical to that characterized by Eqs. (14). The present eigenvalue problem is a realistic one. Here, the shell slides and rotates freely on the constraining surface of conical shape during the entire process of loading and buckling, Fig. 2.

Because of the similarity of their behavior near the edge, the axisymmetric buckling of conical shells, which can slide and rotate freely on the constraining surface of conical shape, can be anticipated to occur at the pressure corresponding to the eigenvalue of the present boundary value problem. The corresponding conical shell is sketched by dashed lines in Fig. 2. This conclusion may provide a supplementary explanation to the result of the recent investigation of Baruch, Harari, Singer.[4] They obtained the low buckling load of the conical shell near $\rho = 0.5$ for the SS3 boundary condition, which turned out to be similar to the free edge for that buckling mode.

$$\begin{aligned} &\text{Case } (iii) \quad Q_e = 0, \quad M_{\phi e} = 0 \\ &\text{Case } (iv) \quad H_e = 0, \quad \beta_e = 0 \\ &\text{Case } (v) \quad u_e = 0, \quad M_{\phi e} = 0 \\ &\text{Case } (vi) \quad u_e = 0, \quad \beta_e = 0 \end{aligned} \tag{20}$$

The result of numerical computations showed that $\rho = 1$ is the only eigenvalue for all these cases of boundary conditions.

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A Direct Modification Procedure for the Displacement Method

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Introduction

The increased speed and capacity of modern computers is permitting the solution of large nonlinear or optimization problems using the Matrix Displacement Method. An efficient modification procedure is an integral feature of the required computer programs. When a relatively small portion of the structure is to be modified at one time, or when convergence difficulties are expected in an iterative approach, a direct method[1–5] can be recommended. In the original Argyris approach,[1,2] the changes in values of the elemental stiffness matrix **k** are interpreted in terms of initial stresses or loads. This method normally involves the triangularization of a matrix of size equal to the number of changed rows in **k** according to the natural element freedoms, and has special application for the elastoplastic problem, as shown in Ref. 3. Sobieszczanski[4] has presented a competitive method for the case of successive independent modifications to a structure. Since the method requires element flexibility matrices, its use in conjunction with existing Displacement Method programs would involve considerable programing effort. Also, this technique is not especially suited to the elastoplastic problem, where all the previously modified members must be modified anew after each load increment.

The size of the modifications matrices can often be considerably reduced if changes are made directly with respect to the global stiffness matrix **K** of the structure. Earlier attempts in this direction presumed that **K** was partitioned into modified and unmodified parts. Even if this transformation

Received February 22, 1971; revision received March 29, 1971.
Index Category: Structural Static Analysis.

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